8.1 Dot Products and Orthogonal Sets

Definition: Suppose that $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are both vectors in $\mathbb{R}^n$. Then the *dot product* of u and v is $u \cdot v = u_1v_1 + \ldots + u_nv_n$.

Theorem: Let u, v, w be in $\mathbb{R}^n$. Then the dot product has the following properties:

- (Symmetry) $u \cdot v = v \cdot u$,
- (Linearity) $(cu + v) \cdot w = cu \cdot w + v \cdot w$,
- (Positive Definite) $u \cdot u \geq 0$ for all u, and $u \cdot u = 0$ if and and only if $u = 0$.

Definition: Let x be a vector in $\mathbb{R}^n$, then the *norm* of x is given by $\|x\| = \sqrt{x \cdot x}$. Note that $\|cx\| = |c|\|x\|$.

For two vectors u and v, the *distance* between u and v is given by $\|u - v\|$.

Definition: Let u and v be vectors in $\mathbb{R}^n$ are *orthogonal* if $u \cdot v = 0$.

Theorem: (Pythagorean Theorem) Suppose that u and v are in $\mathbb{R}^n$. Then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $u \cdot v = 0$.

Theorem: (Triangle Inequality) If u and v are in $\mathbb{R}^n$, then $\|u + v\| \leq \|u\| + \|v\|$.

Definition: Let S be a subspace of $\mathbb{R}^n$. A vector u is *orthogonal* to S if it is orthogonal to every vector in S. The set of all vectors orthogonal to S is called the *orthogonal complement* of S and is denoted $S^\perp$.

The orthogonal complement to a subspace is also a subspace.

Theorem: Let $B = \{v_1, \ldots, v_n\}$ be a basis for a subspace S. Then $u \in S^\perp$ (u is orthogonal to S) if and only if u is orthogonal to each v_i.

Example: Let $s_1 = (1, 0, -1)$ and $s_2 = (1, 1, 1)$ and S be the span of s_1 and s_2. Is $u = (-1, 1, 1) \in S^\perp$? What is a basis for $S^\perp$?

Definition: A set of vectors V in $\mathbb{R}^n$ form an *orthogonal set* the vectors are pairwise orthogonal. This means that if v_i and v_j are distinct vectors in V, then $v_i \cdot v_j = 0$.

Example:

- Is the standard basis an orthogonal set?
- What's a basis that is not orthogonal?

Theorem: An orthogonal set of nonzero vectors is linearly independent.

Definition: A basis that is orthogonal as a set is called an *orthogonal basis*. A basis that is orthogonal as a set and is comprised of vectors of norm 1 is called an *orthonormal basis*.

Theorem: Let S be a subspace with orthogonal basis $\{v_1, \ldots, v_k\}$. Then any vector $s \in S$ can be written as a linear combination $v = c_1v_1 + \ldots + c_kv_k$ with $c_i = v_i \cdot s / \|v_i\|^2$.

Example: (THIS IS A BAD EXAMPLE. TURNS OUT NOT TO BE ORTHOGONAL.) Let $v_1 = (-2, 1, 1), v_2 = (1, -1, -3), v_3 = (4, 7, -1)$. Write $(3, -1, 5)$ as a linear combination of v_i.

For finite dimensional spaces, we have that $(S^{\perp})^{\perp} = S$. Use this to show that every subspace is the null space of some matrix.